What Does Theory Imply About Determinants of The Elasticity of Housing Supply In Cities?

18 March, 2023

Daniel A Broxterman[*]

Yishen Liu[+]

Anthony M. Yezer[#]

Abstract

Studies of the elasticity of housing supply in cities have relied on theoretical results from classical land market models. Unfortunately, past applications of theory have not been correct. This paper corrects problems in previous classical models of housing supply elasticity and adds results for more flexible neoclassical models. Theory implies that housing supply elasticity varies inversely with city size and transportation cost, and directly with cost of structure inputs and rural land. Topographic features and planning regulation do not influence supply elasticity if they are apply uniformly. Finally elasticity is related to the location within the city where housing prices are measured. Overall, these results imply that empirical estimation of determinants of differences in housing supply elasticity across cities is problematic.

JEL Classifications: R14, R31

Key Words: housing supply elasticity, housing price, land use regulation, standard urban model

---

[*]Florida State University, Tallahassee, Florida, USA, <dbroxterman@fsu.edu>

[+]Fannie Mae, Washington, D.C., <yishen_liu@fanniemae.com>

[#]George Washington University, Washington, D.C., yezer@gwu.edu

What Does Theory Imply About Determinants of The Elasticity of Housing Supply In Cities?

I. Introduction

Why does the price elasticity of housing supply vary across cities? This question has been identified important because factors which lower supply elasticity in highly productive cities could have substantial welfare effects.

The empirical literature is divided into studies of short and long run supply elasticities. Short run measures have a putty-clay assumption with supply responses based on new construction on vacant land over a period of one decade. Long run elasticity involves putty-putty adjustments over perhaps 40 years as in Saiz (2010). Long run supply responses include tear downs and rebuilding. Theory allows both short and long run elasticities to be considered in a framework that allows them to be considered and compared.

Given the difficulty of measuring changes in the supply of housing services, the empirical literature has generally measured changes in the number of housing units. Theory allows the derivation of results for short and long run price elasticities of both housing space and units.

Recently Baum-Snow and Han (2022) have demonstrated that the elasticity of housing supply varies across city neighborhoods. The theory presented here shows that distance from the CBD should influence measures of the elasticity of long run housing supply. However, short run housing elasticity variation is likely due to local variation in land availability that are beyond the models used here.

Theory allows the effect of variation in measures of housing price to be compared. It is possible to measure price change at a fixed location, the CBD, the location of the housing unit at the mean distance from the CBD or the difference between prices of existing and newly constructed units. Finally, the theoretical analysis is performed using change in rental prices that reflect the use value of housing rather than confusing use and option value by relying on changes asset prices. Carrillo, Harris, and Yezer, (2023) show that the standard deviation in asset prices across cities is twice the standard deviation in rental prices.

Two theoretical approaches to modeling the urban housing market are considered here. First is the classical model in which there is one identical house per unit land and homogenous households consume that single unit regardless of price or location within the city. The classical model is important because it has been used in Saiz (2010) and Green, Malpezzi, and Mayo (2005) to justify empirical testing. Second, is the neoclassical model in which housing is produced using land and structure inputs that are substitutable and households maximize utility subject to a price per unit housing space and commuting cost or equivalently wages that vary with distance to the center city. Cosman, et. al. (2018) have used a different version of the neoclassical model to develop implications for short run supply elasticity.[1] Both classical and neoclassical models are solved for the theoretical determinants of the price elasticity of housing supply under alternative definitions and implications of the results for empirical testing are developed. In addition to demonstrating that past analysis of the classical model has produced misleading implications for the elasticity of housing supply, the theory demonstrates that elasticity varies with housing price and city size. The effects of other factors on housing supply elasticity act through their influence on the relation between housing price and elasticity. This

[1] Several of the arguments made here are logically consistent with Cosman, et. al. (2018) although the model of housing supply in cities is quite different. Their model requires housing to be supplied on previously undeveloped land whereas the long run model here assumes that land used for housing can be redeveloped at a higher density and specific functional forms for the housing production function and housing price gradient are not assumed.

creates fundamental problems in attempts to create consistent measures of the determinants of housing supply elasticity in any given city. It also shows why current approaches to estimating housing supply elasticity are likely to produce biased and misleading estimates of the determinants of intercity variation in housing supply elasticity.

This theoretical discussion has implications for attempts to estimate determinants of housing supply elasticity. These implications are distinct from econometric issues that Davidoff (2016) has noted regarding identification of the supply function that are not considered here. In addition, Goodman (2013) has demonstrated substantial non-linearity in the empirical housing supply schedule of cities which implies that supply is hysteretic. This view has been confirmed recently by Aastveit and Anundsen (2022) who find that measured supply elasticity depends on the size and sign of the shock to each city. This paper abstracts from problems of urban amenity and urban decline. Instead the supply elasticity that would be expected in an continuously growing city is being considered here.

The next section of this paper uses a classical urban housing market model to analyze the effects of constraints on land availability for housing, limits on structure density, transportation cost, city size, cost of structure inputs, rural land price, and structure input costs on supply elasticity. The classical model has been used in the past to develop implications for housing supply but the results obtained here are new and different. In a classical model, the supply elasticity of housing units and housing space and long and short run elasticities are identical. In the next section, a neoclassical urban land market model is used to determine factors that alter supply elasticity. In a neoclassical model, short and long run elasticities differ and the elasticity of supply of housing units and housing space must be distinguished. Overall the results of this theoretical inquiry suggest that many factors neglected in past empirical research, including transportation cost, city size, rural land price, and structure input costs have important effects on the elasticity of housing supply. However, the fraction of land available for residential real estate and uniform limits on density, factors commonly associated with topography or planning regulation, do not influence housing supply elasticity.

II. Housing Supply Elasticity in a Classical City

As noted above, there are two approaches to modeling the price elasticity of housing supply. The long run assumption of putty-putty construction assumes that, in response to a change in price, the entire housing stock is subject to modification. Additional housing space may be built on land currently occupied by existing units or on undeveloped land. In contrast a short run assumption of putty-clay housing only allows construction on formerly vacant land, presumably converted from non-urban uses at the city edge. In the case of a classical city, the Leontief production function fixes the housing to land ratio regardless of housing price. Accordingly the only possibility for adding housing space or units is conversion of land at the edge of the city. Therefore, the results presented here hold for both long and short run supply responses to changes in the price of housing.

The distinguishing characteristic of a classical city is that households consume a standard quantity of housing, h, and the housing production function is Leontief, so that:

$$H = Min\,[\alpha l, \beta s] \qquad \text{(II-1)}$$

where H is the quantity of housing space, l is land, s is structure inputs, and α and β are parameters reflecting output per unit input. Producers of housing will set $\alpha l = \beta s$ so that housing output is a simple multiple of land, $H = \alpha l$. Classical households consume a fixed amount of housing, h, which is normalized to unity in order to economize on notation. The housing producer's cost function at any location is:

$$C = rl + is \qquad \text{(II-2)}$$

where r is the rental price of land at that location and i is the rental price of structure inputs which is assumed invariant. If competition forces developers to set price equal to average cost then:

$$C/H = rl/H + is/H = rl/\alpha l + is/\beta s = r + i = p \qquad \text{(II-3)}$$

with p equal to the rental price of a housing unit because H is the number of housing units supplied.

Households either must commute to the city center or earn wages that are lower than center city wages by the amount of transportation costs to the center. Letting p_o be price of housing units at the city center, and k indicate distance to that center, the rental prices of housing and land at distance k are given by:

$$p_o = r_o + i, \quad p_k = p_o - tk, \ \text{and} \ r_k = p_k - i = p_o - tk - i \qquad \text{(II-4)}$$

Differentiation of (II-4) with respect to k yields $dp/dk = -t$ which is a classical version of Muth's equation under the assumption that housing consumption is constant and equal to unity.

If the city limit is at k^*, p^* is housing price at that limit and the opportunity cost of urban land, commonly called the rental price of agricultural land, is r_A, then:

$$r_A = r_* = p_* - i = p_o - tk^* - i \ \text{ or } k^* = (p_o - p^*)/t \qquad \text{(II-5)}$$

Housing supply, both space and units, is proportional to land in the city used for housing. Letting Λ equal the fraction of land available for residential construction, the total supply of both city housing, H, and number of units, N, is given by:

$$H = \int_o^{k^*} (\Lambda 2\pi\alpha)\, k\, dk = (\Lambda\pi\alpha)\, k^{*2} \qquad \text{(II-6)}$$

While the assumptions of the classical model may appear to depart from reality, thus far the empirical literature on housing supply elasticity has appealed to classical models to support stochastic specification of equations used to estimate supply elasticity.[2]

The influence of planning or topography on housing supply elasticity is reflected in two parameters of the classical model. First limits on building density, if binding, tend to lower α by requiring more land per structure than would be required given the production technology. Similarly, topography may limit construction density and lower α. Second Λ may be reduced by planning regulations that require open space or otherwise limit the fraction of land available for residences or by constraints imposed by topography on buildable land. Therefore analyzing the effects of planning restrictions or topography on the elasticity of housing supply, requires that the role of these parameters in this theoretical model of housing supply elasticity be determined.

Proposition I: In a classical urban land market the elasticity of supply of housing units and space in both the short and long run varies inversely with city population and transportation cost and directly with cost of agricultural land and structure inputs. Supply elasticity is not determined by the parameters Λ and α which represent factors such as topography and building regulations that limit the fraction of land available for housing or the density of housing units.

[2] For example see Saiz (2010) and Green, Malpezzi, and Mayo (2005) who appeal to versions of the classical model.

The starting point in exploration of the elasticity of housing supply is to define the term. In the classical model all housing units are the same size and the price per square foot is directly proportional to the price per housing unit. The notation E will indicate elasticity based on housing units and ϵ the elasticity based on housing space. In the classical model considered here $E = \epsilon$. The percentage change in rental price of housing services may depend on location.[3] Initially consider price at the city center, p_o. This definition is appealing because rents at the city center do not require a measure of commuting cost. Accordingly the initial definition of housing supply is $\epsilon = dlog\ H/dlog\ r_o$.

For a circular city with proportion Λ of land available for housing, this is easily expressed by taking the logarithm of (II-6):

$$log\ H = log\ \Lambda + log\ \pi + log\ \alpha + 2\ log\ k^* \qquad \text{(II-7)}$$

Where k is distance from the city center and k^* is the city limit. Given that households are mobile, there is an iso-utility condition that requires the rental price of housing units decline with increasing distance according to (II-5).

The elasticity of the city limit with respect to center city price is found by using (II-5):

$$\partial log\ k^*/\partial log\ p_o = (\partial k^*/\partial p_o)\ (p_o/k^*) \quad or$$

$$= (p_o/t)/\ [p_o\ /(p_o - p^*)/t] = p_o\ /(p_o - p^*) > 0 \qquad (II\text{-}8)$$

[3] The annual cost of housing, which determines demand, is based on rental price. However, supply is based on the asset price, which requires knowledge of the capitalization (cap) rate. In the discussion here, variation in the cap rate both within and across cities will be ignored. However, this is a very consequential issue for empirical estimation of housing supply elasticity.

This implies that the elasticity of city radius with respect to central rent is not constant and is only a function of central rent and $p^* = r_A + i$. Specifically, the change in elasticity of the city radius with respect to rent, $d^2 log\ k^*/dlog\ p_o^2 = -p^*/(p_o - p^*)^2 < 0$. While the positive sign of the first derivative of $log\ k$ with respect to $log\ p_o$ is not surprising both the fact that the size of the derivative varies with p_o and p^* and the negative sign of the second derivative are less intuitive.

Continuing the focus on the elasticity of housing supply defined as $\epsilon_{po} = dlog\ H/dlog\ p_o$, totally differentiating (II-7) yields:

$$dlog\ H = dlog\ \Lambda + dlog\ \pi + dlog\ \alpha + 2\ dlog\ k^* \qquad \text{(II-9)}$$

Therefore the elasticity of housing supply as a function of p_o can be written as:

$$\epsilon_{po} = dlog\ H/dlog\ p_o = (dlog\ \Lambda/dlog\ p_o) + (dlog\ \pi/dlog\ p_o)$$

$$+ (dlog\ \alpha/dlog\ p_o) + 2\ (dlog\ k^*/dlog\ p_o) \qquad (II\text{-}10)$$

Clearly this reduces to:

$$\epsilon_{po} = dlog\ H/dlog\ p_o = (dlog\ k^*/dlog\ p_o) = 2[p^*/(p_o - p^*)^2] > 0 \qquad (II\text{-}10')$$

First, given that p_o has no effect on the parameters Λ, π, or α in the classical model, this establishes the second part of *Proposition I*. Under a straightforward definition of elasticity of supply based on p_o, changes in Λ and α, the two parameters reflecting the effects of planning or topography have no effect on housing supply elasticity. Second, the process of city growth in a classical model necessarily involves an increase in p_o. This means, as stated in the second part of *Proposition I* that, in classical model, housing supply elasticity, defined as $\epsilon_{po} = dlog\ H/dlog\ p_o$, is not constant. Instead, it varies with the level of p_o, just as $dlog\ k^*/dlog\ p_o$ varies with p_o. Holding p^* constant, elasticity falls as p_o rises with city size, i.e. $\partial\epsilon_{po}/\partial p_o < 0$. This is a natural result of the geometry of cities, the definition of elasticity based on p_o, and the response of construction to the excess of rent above that required for rural land conversion and construction. Finally, the two parameters, r_A and i, that raise p^*, *(II-5)* $p^* = r_A + i$, raise the elasticity of housing supply as seen by differentiating *(II-10')* yielding $\partial\epsilon_{po} /\partial p^* = -(p_o + p^*)/(p^* - p_o)^3 > 0$. This counterintuitive result arises because, as p_o rises, holding p_o constant, city population falls and that smaller size results in higher price elasticity of supply.

The final element of *Proposition I* concerns the effect of transportation cost, t, on housing supply elasticity. This can be seen by from *(II-10')* where $\epsilon_{po} = dlog\ H/dlog\ p_o = 2[p^*/(p_o - p^*)^2]$ and noting from *(II-5)* that $(p_o - p^*)$ varies directly with t.

The results comprising *Proposition I* contrast with the assumption in the empirical literature that the elasticity of housing supply is not a function of city size, that transportation cost, can be omitted from estimates of supply elasticity and that differences in Λ *or* α, due either to topography or land use planning cause differences in supply elasticity. The results here imply that no such relations exist provided that Λ and α are not a function of k.[4] There is a further counterintuitive implication of *(II-10')*. To the extent that regulation or higher construction costs raise p^* by raising i or r_A, the elasticity of housing supply will rise. These results hold for the relation between the percentage change in rental price at the city center and the percentage change in the number of housing units or the amount of housing space because these two are identical in the classical model.

Thus far the analysis has been conducted in terms of price at the CBD, p_o. This is potentially easily observed, particularly as a rental price, in empirical work. However prices can

---

[4] In the case of many cities Λ and α likely vary with k. However the pattern of that variation is far from uniform and may be either increasing or decreasing as a function of k.

be observed at alternative locations. The next proposition extends the analysis to elasticity measures using prices measured at distances ranging to the city edge. The first two propositions apply to these prices also and, in addition, elasticity varies inversely with distance. This means that finding some way to control the distance at which price change is measured empirically within cities over time or across cities appears to pose an additional challenge for empirical estimation of housing supply elasticity.

Proposition II: Housing supply elasticity falls with the distance from the CBD at which the price change is measured.

An alternative definition of the elasticity of housing supply might choose rent changes at some other location, e.g. $r^{\#}$, *where* $0 < k^{\#} < k^{*}$. The effect of this switch on the elasticity of supply is straightforward. Given that $p_o = p^{\#} + tk^{\#}$, and $dp_o/dp^{\#} = 1$, it follows that

$$dlog\ p_o/dlog\ p^{\#} = dp_o/dp^{\#}\,(p^{\#}/p_o) = p^{\#}/(p^{\#} + tk^{\#}) > 0 \text{ and } < 1 \qquad (II\text{-}12)$$

and $\epsilon_{p\#} = dlog\ H/dlog\ p^{\#} = (dlog\ H/dlog\ p_o)(dlog\ p_o/dlog\ p^{\#})$

$$= 2[p^{*}/(p_o - p^{*})^2]\,[p^{\#}/(p^{\#} + tk^{\#}/h)] \qquad (II\text{-}13)$$

Hence $\epsilon_{|p\#} = dlog\ H/dlog\ p^{\#}$ is equal to the product of $dlog\ H/dlog\ p_o$ and a term, $0 < [p^{\#}/(p^{\#} + tk^{\#}/h)] < 1$, which varies inversely with distance, $k^{\#}$. Housing supply elasticity measured by rents at a distance $0 < k^{\#} < k^{*}$ is uniformly smaller than supply elasticity measured by p_o and it decreases monotonically with distance $k^{\#}$. Thus all of the problems with defining and measuring the elasticity of housing supply in cities that occur when constraining rental price to the city center also arise when price is measured at any other fixed radius between the center and edge. Furthermore changing the location at which price is measured, changes the elasticity of housing supply.

For empirical purposes it might be attractive to use the average rent of all units in the city. The average location of a housing unit, noted $k^{@}$, is determined by:

$$k^{@} = [\int_o^{k^*} (\Lambda 2\pi/\alpha)\ k\ k\ dk\ ]/[\int_o^{k^*} (\Lambda 2\pi/\alpha)\ k\ dk] = (2/3)k^* \qquad \text{(II-9)}$$

Unfortunately relying on mean rent does not solve the problems with rent at other locations. Setting $k^{\#} = (2/3)k^*$ does not remove any of the issues associated with a fixed location rent at $k^{\#}$.

Taken together these considerations make the relation between percentage change in housing units or services and percentage change in the average housing unit rent, even if this could be measured empirically, truly problematic as a measure of the elasticity of housing supply in a classical urban land market model.

Note that, because both long and short run supply responses only occur on previously vacant land at the city edge the conclusions regarding determinants of housing supply elasticity in this section apply to both long and short run supply models. Supply elasticity falls with city size and transportation cost and rises with the agricultural reservation price and cost of structure inputs but does not depend directly on the Λ, α, or t parameters. Furthermore, supply elasticity falls with the distance from the city center at which housing price is measured.

III. Housing Supply Elasticity in a Neoclassical City

III-1. Long Run (putty-putty) Housing Supply Elasticity

In a neoclassical model, long run (putty-putty) supply elasticity differs from short run (putty-clay) elasticity because housing price increases result in additional housing space construction throughout the city. Indeed, vacant land conversion may constitute a small portion of the long run supply response. Given the durability of housing, this process may take several decades which creates significant empirical issues that need not be confronted in a theoretical model.

Also, in a neoclassical model, households substitute away from housing consumption when its relative price rises and the size of housing per household falls.[5] Thus supply elasticity of space differs from supply elasticity of housing units. Results for both long run elasticities in a putty-putty are developed here. This is a theoretical exercise. It does not claim to determine the time frame needed to achieve a long versus short run equilibrium adjustment of the housing market. Indeed, given that housing prices are changing continuously, it is not clear that a long run equilibrium of the housing market is ever observed empirically.

The setup here follows Brueckner (1987). Developers produce housing according neoclassical production function, Q = F(K, l), where K and l are structure and land, respectively and F(.) is concave in inputs with constant returns to scale. The developer's problem is to choose inputs, K and l, to maximize profit:

$$Max_{K,l} \quad p_k F(K, l) - iK - r_k\, l \qquad (III\text{-}1)$$

where p_k is the rental price per unit housing space at distance k from the CBD, i is the exogenous market price of capital, and r_k is land rent. Land is assumed owned by absentee landlords. Entry of developers drives their economic profits to zero.

Given constant returns to scale, it is possible to write the developer's problem in terms of profit per unit land and housing production in terms of the $s = K/l$ ratio. Housing production per

[5] The analysis only concerns consumption of a primary residence. Consumption of a secondary residence is considered to be part of the composite commodity, having nothing to do with the local housing market being modeled.

unit land is given by $h(s) = F(s, 1)$, where $h'(s) > 0$ and $h''(s) < 0$. The developer's problem is now to maximize $p_k h(s) - is - r_k$. First-order conditions for zero profit equilibrium yield:

$$p_k \partial h/\partial s = i \qquad \text{(III-2)}$$

$$p_k h(s) = is + r_k \qquad \text{(III-3)}$$

This system yields solutions for optimal structure to land ratio, $s = S(p, i)$ and land rent $r = R(p, i)$.

Proposition III: Supply elasticity is not determined by the fraction of land available for housing production, Λ, which may reflect factors such as topography and building regulations that limit the fraction of land available for housing.

Total housing production at a given distance, k, is given by

$$H_k = 2\pi\Lambda k\, h(s_k) = 2\pi\Lambda k\, h(S(p_k, i))) \qquad \text{(III-4)}$$

It follows that, as shown for the classical model and equations *(II-9 through II-10')*, this elasticity is not a function of Λ, *i.e.* $\partial \epsilon_{|k}/\partial\Lambda = 0$. Here Λ can be interpreted as the fraction of land available for development due to topography, regulation or preemption by non-residential land uses. As with the classical model, this elasticity is not a function of Λ, *i.e.* $\partial \epsilon_{|k}/\partial\Lambda = 0$. *Proposition III is a restatement of a portion of Proposition I* and demonstrates that the lack of a relation between Λ and supply elasticity holds for neoclassical as well as classical models as it relies on a fundamental geometric property of the urban land market.

For given k the price elasticity of housing supply is

$$\epsilon_{|k} = [\partial H_k/\partial p_k][p_k/H_k] = [\partial \log h(S(p_k, i))/\partial p_k]\, p_k \qquad \text{(III-5)}$$

Because this is a neoclassical model, the elasticity of housing space supply is not equal to the elasticity of housing unit supply. Households respond to rising price by consuming less space. Let h_k be housing space per household at distance k. Then households at that distance are given by $N_k = H_k/h_k$ and changes in this household count are related to changes in housing space by:

$$dN_k = [\partial(H_k/h_k)/\partial H_k]\ dH_k + [\partial(H_k/h_k)/\partial h_k]\ dh_k \qquad (III\text{-}6)$$

The effect of house price on households is:

$$dN_k/dp_k = [(1/h_k)\ \partial(H_k/\partial p_k)] - [(H_k/h_k^2)[\partial h_k/\partial p_k)] \qquad (III\text{-}6')$$

Therefore the elasticity of supply of housing units can be written as:

$$E_k = \{[(1/h_k)\ \partial(H_k/\partial p_k)] - [(H_k/h_k^2)[\partial h_k/\partial p_k)]\}\ [p_k/(H_k/h_k)] \quad or$$

$$E_k = [\partial(H_k/\partial p_k)(p_k/H_k)] - [\partial h_k/\partial p_k)](p_k/h_k) \qquad or$$

$$E_k = \epsilon_k - [\partial h_k/\partial p_k)](p_k/h_k) = \epsilon_k - \varepsilon_k \qquad (III\text{-}6'')$$

Where $\varepsilon_k < 0$ is the own price elasticity of household demand for space at k. Clearly $E_k > \epsilon_k > 0$.

Now consider total city housing space supply from $k = k_o$ to the city boundary at k^* where the rental price of urban land falls to the agricultural reservation price, $r_A = R(p_{k*}, i, k^*)$. Total housing production is

$$Q = \int_o^{k*} 2\pi\Lambda\ h(S(P(k), i)\ dk \qquad (III\text{-}7)$$

Let $p^@$ be the average price of housing in the city and hence it is the price at which the average density of housing is produced $h^@ = h(S(p^@, i)$.[6] With this definition, city space supply elasticity can be written as:

[6] Proof of the existence of such an average price is given in the appendix.

$$\epsilon = [\partial Q/\partial p^{@}]\ [p^{@}/Q] = [\partial \log h(S(p^{@}, i)/\partial\ dp^{@}]\ p^{@} \qquad (III\text{-}8)$$

Once again this expression is not a function of Λ, i.e. $\partial\ \epsilon/\partial\Lambda = 0$.

This was also noted for housing supply elasticity in the classical model. Similarly housing unit supply, $E = \epsilon_k - \varepsilon_k$, is not a function of Λ, i.e. $\partial E\ /\partial\Lambda = 0$. This is in contrast with previous literature which has argued that housing unit supply elasticity depends on topography, regulation, or other factors influencing the fraction of land available for residential real estate.

Proposition IV: Supply elasticity of both interior space and number of units varies inversely with transportation cost, t.

This section considers long run effects of changes in transportation cost on the elasticity of supply of housing space and units.

Homogenous households maximize utility by choosing a composite non-housing good with price equal unity everywhere and housing space, h, subject to a budget constraint

$$Max_{c,h}\ U(c, h)$$

$$s.t. \quad y = c + tk + P(k)h \qquad (III\text{-}9)$$

where t is a uniform commuting cost per unit distance, and y is exogenous income earned at the city center. Workers employed outside the center at distance k earn $y - tk$. Muth's equation derived from the household's problem in *(III-9)* implies that $\partial P(k)/\partial k = -t/h$ and it follows that $\partial P(k)/\partial t < 0\ k \leq k^*$. In equilibrium, Equation (III-5), supply elasticity of housing space at any k can be written:

$$\epsilon_{|k} = [\partial \log h(S(p_k, i))/\partial p_k]\ p_k = [h'(s)/h(s)][\partial s/\partial p_k]\ p_k$$

$$= [i/h(s)][\partial s/\partial p_k] \qquad (III\text{-}10)$$

where the third equality follows directly from the first-order conditions for the developer's optimal choice.

Finally, to determine the effects of variation in transportation cost, differentiate Equation (III-10) with respect to transportation cost.[7] With some algebraic manipulation the derivative of housing space elasticity at any given distance with respect to transportation cost can be written as:

$$\partial \epsilon_{|k}/\partial t = [i/h(s)][\partial p_k/\partial t] \{[h'(s)/h(s)][\partial s/\partial p_k]^2 - [\partial^2/s/\partial p_k^2]\} \qquad (III\text{-}11)$$

The product, $[i/h(S)][\partial p_k/\partial t]$, is clearly < 0. This yields the initial result that unless the expression in brackets, { }, is zero, housing supply elasticity at any given distance varies inversely with transportation cost. The first term of the expression in brackets is clearly positive, $[h'(S)/h(S)][\partial S/\partial p_k]^2 > 0$. Under the usual assumptions regarding the housing production function, $[\partial^2/s/\partial p_k^2] < 0$ and the effect of increasing transportation cost on the elasticity of housing space supply at any distance is negative.

Effects of transportation cost on the elasticity of supply of housing units follow from (III-6") by subtracting own price elasticity of demand from the expression for $\epsilon_{|k}$ in (III-10):

$$E_{|k} = \epsilon_{|k} - \varepsilon_{|k} = [i/h(s_k)] [\partial s_k/\partial p_k] - [\partial h_k/\partial p_k](p_k/h_k) \qquad (III\text{-}12)$$

The derivative of housing unit supply elasticity at any distance *k* with respect to the transportation cost parameter is given by:

$$dE_{|k} /dt = [i/h(s_k)][\partial p_k/\partial t] \{[h'(s_k)/h(s_k)][\partial s_k/\partial p_k]^2 - [\partial^2/s_k/\partial p_k^2]\}$$

[7] Transportation cost enters the supply elasticity equation through its effect on the consumer's optimization problem, i.e. through Muth's equation.

$$-\ [\partial h_k/\partial p_k]\{(\partial p_k/\partial t)/h_k - (p_k/h_k^2)(dh_k/dt)\} \qquad (III\text{-}13)$$

The first expression in brackets is just $d\epsilon_{|k}/dt < 0$. Given that $[\partial h_k/\partial p_k] < 0$, $(\partial p_k/\partial t) < 0$, and $(dh_k/dt) > 0$, the entire expression $[\partial h_k/\partial p_k]\{(\partial p_k/\partial t)/h_k - (p_k/h_k^2)(dh_k/dt)\} > 0$ and, it follows that $dE_{|k}/dt < d\epsilon_{|k}/dt < 0$. Increases in the transportation cost parameter, t, reduce the elasticities of both housing space and housing unit supply at a given location, k.

The discussion of the effects of transportation cost on the overall elasticity of supply can be extended to the elasticity of supply for the entire city. As was the case in the previous section, simply apply the argument in which p_k is replaced by average price, $p^{@}$.

III-2. Short Run (putty-clay) Supply Elasticity in a Neoclassical City

The short run, putty-clay, response of housing in a neoclassical city has much in common with that in a classical city because developers are only able to be active on the vacant land at the edge of the city. The supply response of space and units is different because neoclassical households substitute away from housing when its prices rise and unit sizes shrink. This adjustment in unit size will be assumed possible for all housing in the short run. The same arguments that motivate Proposition IV may be applied here to prove that the short run elasticity of housing supply is not a function of, Λ, the fraction of land available for housing.

Similarly, the density of housing added in each successive annulus as the price of housing rises, is identical because the developer's optimal solution at the city edge is always identical. This means that the entirety of Proposition I, applies, except that the parameter "α" is interpreted as the structure land ratio at the city edge.

Proposition V: In the short run of a neoclassical (putty-clay) urban land market the elasticity of supply of housing space in the short run varies inversely with city population and *transportation cost and directly with cost of agricultural land and structure inputs. Supply elasticity is not determined by the parameters Λ and α which represent factors such as topography and building regulations that limit the fraction of land available for housing or the density of housing units.*

Short run elasticity of supply of housing units is really annoying because as city radius expands, the ratio of existing to new units rises and the difference between elasticity of supply of space and of units increases with that radius. Proof of this proposition coming.

IV. Conclusions and Implications

Empirical estimates of the relation between changes in housing price and housing units in growing cities provide some insight it the elasticity of housing supply. A recent example by Accetturo, et. al. (2021) finds that estimated supply elasticity varies substantially among Italian cities and that future rates of city growth vary inversely with the differences in these measured elasticities. This research is not designed to cast doubt on such results beyond noting that results may be sensitive to the manner in which housing price change is measured and identification in such research can be an issue. The substantial differences measured supply elasticities across cities are consistent with the predictions of the theory presented here in a world where the issues of nonlinearity in supply in declining cities raised by Goodman (2015) are not a concern.

However, the theory does have major implications for attempts to explain the reason for differences in estimated supply elasticity among cities. In particular, some factors, such as topographic barriers, and restrictions on height and density are shown to have a problematic relation to supply elasticity unless they distributed systematically unevenly over space. Furthermore a list of factors that are very difficult to measure, including transportation cost and value of land at the urban edge, are consequential for supply elasticity. This means that it is very easy to confuse the importance of various factors influencing housing supply elasticity. Finally there is a tendency for supply elasticity to fall with city size that confounds attempts to measure determinants of housing supply itself.

Certainly the elasticity of housing supply in cities is consequential because it determines the allocation of labor among more or less productive locations. The theory presented here suggests that lowering urban transportation costs can raise the elasticity of housing supply Formulating other policies that address low housing supply elasticity, based on empirical estimates of determinants of the relation between city characteristics and measured supply elasticity may be problematic.

References

Aastveit, Knut Are, Andre K. Anundsen. (2022). Asymmetric effects of monetary policy in regional housing markets, *American Economic Journal: Macroeconomics*, 14(4), 499-525.

Accetturo, A, A. R. Lamorgese, S. Mocetti, and D. Pellegrino. (2021). Housing supply elasticity and growth: evidence from Italian cities. *Journal of Economic Geography*, 21(3), 367-396.

Baum-Snow, Nathan, and Liu Han. 2019. The microgeography of housing supply.

Cosman, J., T. Davidoff, and J. Williams. (2018) Housing appreciation and marginal land supply in monocentric cities with topography.

Davidoff, Thomas, 2016. Supply constraints are not valid instruments for home prices because they are correlated with many demand factors, *Critical Finanac Review*, 5(2) 177-206.

DiPasquale, Denise. 1999. Why don't we know more about housing supply? *Journal of Real Estate Finance and Economics*, 18 (1), 9-24.

Fischer, Manfred M., Florian Huber,Michael Pfarrhofer, and Petra Staufer-Steinnocher, 2021. *Real Estate Economics*, 49(4), 1039-1068.

Goodman, Allen C. 2013. Is there an S in urban housing supply? Or whatever happened in Detroit? *Journal of Housing Economics*, 22 (3), 179-191.

Green, Richard K., Stephen Malpezzi, and Stephen K. Mayo. 2005. Metropolitan-specific estimates of the elasticity of housing supply and their sources, *American Economic Review*, 95 (2) 334-339.

Howard, Greg, and Jack Liebersohn, 2021. Why is the rent so darn high? The role of growing demandto live in housing-supply-inelastic cities. *Journal of Urban Economics*, 124, 1-14.

Huang, Wenbin, 2022. Government land regulations and housing supply elasticity in urban China, *China and World Economy*, 30(4), 122-148.

Monte, F., S. Redding, and E. Rossi-Hansburg. (2018). Commuting, migration, and local employment elasticities, *American Economic Review*, 108(12), 3855-3890.

Saiz, A. (2010). The geographic determinants of housing supply. *The Quarterly Journal of Economics*, 125(3), 1253–1296.