Abstract

When agents can enter into private trades and contracts are non-exclusive, the incentive efficient allocation cannot be implemented in an adverse selection insurance economy. Optimal taxation is studied under the assumptions that contracts are non-exclusive and private trading cannot be prevented. The existence of a competitive equilibrium, which can be problematic in these economies, is addressed by first deriving a set of feasible allocations such that an equilibrium exists and then determining the optimal allocation within this set. The approach is based on Golosov and Tsyvinski (2007) and Farhi, Golosov, and Tsyvinski (2009) using the model in Labadie (2009).

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When agents can enter into private trades and contracts are non-exclusive, the incentive efficient allocation cannot be implemented in an adverse selection insurance economy. Optimal taxation is studied under the assumptions that contracts are non-exclusive and private trading cannot be prevented. The existence of a competitive equilibrium, which can be problematic in these economies, is addressed by first deriving a set of feasible allocations such that an equilibrium exists and then determining the optimal allocation within this set. The approach is based on Golosov and Tsyvinski (2007) and Farhi, Golosov, and Tsyvinski (2009) using the model in Labadie (2009).

The basic model is described in section 1. The incentive-efficient allocation is derived in the next section and the implications of retrading and nonexclusivity are explored in section 2. The market structure is discussed in section 3, in particular developing the argument that, if there is any trade in side markets, then all agents will face the same contingent claims prices. The derivation of the anonymous equilibrium is in section 4, with the proof of existence in the appendix, and conditions for the existence of an equilibrium are provided. The social planner’s problem with retrading is described next. Finally a closed form solution is derived for logarithmic utility.

1 Basic Model

The basic model is the adverse selection insurance economy of Rothschild and Stiglitz [10] and Prescott and Townsend [9].

This is a single-period pure endowment economy with a single consumption good. There is a continuum of agents indexed over the unit interval and two types of agents, $a$ and $b$. A fraction $f_a$ of agents are type $a$ and $f_b = 1 - f_a$ are type $b$. An agent’s type is private information.

The endowment $\theta$ is a discrete random variable that takes two values $0 \leq \theta_1 < \theta_2$, so that $\theta_1$ is the “bad” state and $\theta_2$ is the “good” state. The random variable $\theta$ is independently distributed across agents. For a type $\eta$ agent, the probability of drawing $\theta_i$, $\eta \in \{a, b\}$ and $i \in \{1, 2\}$. Let $g_{a2} > g_{b2}$ so that type $a$ agents are “low” risk and type $b$ are “high” risk. Denote

$$R_\eta \equiv \frac{g_{\eta i}}{g_{\eta 2}}$$

for $\eta \in \{a, b\}$ as a measure of type risk. Observe that $R_b > R_a$. For any agent, the realization of $\theta$ is public.
information.

Let
\[ \bar{\theta}_\eta = g_\eta_1 \theta_1 + g_\eta_2 \theta_2 \quad \eta = a, b \]
denote the expected endowment for type \( \eta \), where \( \bar{\theta}_a > \bar{\theta}_b \), and let
\[ \bar{\theta} = f_a \bar{\theta}_a + f_b \bar{\theta}_b \] (1)
denote average endowment.

The following assumption is standard. It says that, for either type of agent, the probability of being in any endowment state is strictly positive.

**Assumption 1** \( g_\eta_i > 0, \eta = a, b, i = 1, 2 \).

A type-\( \eta \) agent has preferences
\[ \sum_i g_\eta U(c_i). \] (2)

**Assumption 2** The function \( U \) is continuous and twice continuously differentiable, strictly increasing and strictly concave. Also, as \( c \to 0 \), \( U'(c) \to \infty \) (the Inada condition).

The expectation in (2) uses an individual agent’s conditional probability of realizing \( \theta \). Since there is no aggregate uncertainty, the only risk an agent faces is his idiosyncratic endowment risk.

The realization of an agent’s endowment is public information. A consumption allocation for a type \( \eta \) agent is \( c(\eta) = (c_1(\eta), c_2(\eta)) \) and satisfies a non-negativity constraint. Let
\[ C \equiv \{ c(\eta) = (c_1(\eta), c_2(\eta)) \mid c_i(\eta) \geq 0, i = 1, 2 \}. \]

A pair of consumption allocations \( (c(a), c(b)) \in C^2 \) is feasible in the aggregate if the economy-wide resource constraint holds. Since there is a continuum of agents, the Law of Large Numbers implies that the set of consumption allocations that are feasible in the aggregate is
\[ F \equiv \left\{ (c(a), c(b)) \in C^2 \mid \bar{\theta} \geq \sum_\eta \sum_i f_\eta g_\eta c_i(\eta) \right\}. \] (3)

**DEFINITION:** The pair of consumption allocations \( c \in F \) is incentive compatible if
\[ \sum_i g_\eta U(c_i(\eta)) \geq \sum_i g_\eta U(c_i(h)), \quad h \neq \eta, \quad h, \eta = a, b. \] (4)

Let \( I \subset F \) denote the set of feasible consumption allocations that are incentive compatible.
2 Incentive Efficiency and Private Trading

When private trading among agents can be prevented, the social planner can determine the state-contingent consumption of all agents and is able to implement an incentive-efficient allocation.

**Definition:** The consumption allocation \( c \in I \) is **incentive efficient** if there is no other pair \( \hat{c} \in I \) such that

\[
\sum_{i} g_{\eta i} U(\hat{c}_{\eta i}) \geq \sum_{i} g_{\eta i} U(c_{\eta i}), \quad \eta = a, b,
\]

with strict inequality for at least one type.

Incentive-efficient allocations have been studied extensively by Prescott and Townsend, among others. As demonstrated by Prescott and Townsend, there are three types of incentive-efficient allocations for this economy: (i) full insurance for type \( a \) and partial insurance for type \( b \) (separating); (ii) full insurance for type \( b \) and partial insurance for type \( a \) (separating); (iii) full insurance for both such that \( c_{\eta i} = \bar{\theta} \) (pooling).

Let \( \bar{c}_{\eta} \) denote the certain consumption of the type with full insurance. For the type-\( h \) agent with partial insurance, \( h \neq \eta \), the consumption allocation satisfies

\[
U(\bar{c}_{\eta}) = g_{h1} U(c_{h1}) + g_{h2} U(c_{h2}),
\]

\[
\bar{\theta} - f_{\eta \bar{c}_{\eta}} = f_{h} [g_{h1} c_{h1} + g_{h2} c_{h2}].
\]

Let \( \hat{c}_{h} = (\hat{c}_{h1}, \hat{c}_{h2}) \) denote a solution to (6)-(7). The solution may require transfers across types if \( \hat{c}_{\eta} \neq \bar{\theta}_{\eta} \).

The decentralization of this constrained Pareto-efficient allocation has been the focus of many papers, including Prescott and Townsend. To decentralize an incentive-efficient allocation requires that markets are separated, private trading is prohibited, and contracts are exclusive. Agents can enter only one market (separation) and one contract (exclusivity), and must consume the quantities specified in the contract (no private trading). A detailed description is contained in Bisin and Gottardi [2].

**Private Trading and Incentive Efficiency**

Suppose that the social planner chooses an incentive-efficient allocation where only one type of agent receives full consumption insurance (separating allocation) and that private trading among agents cannot be
prevented. That is, after an agent self-selects into a market and purchases an insurance contract, but before observing the realization of his endowment, agents can enter into risk-sharing arrangements with other agents. Agents may desire to enter into risk-sharing arrangements if it enables them to effectively “unbundle” the contingent claims in a contract.

For simplicity, focus on the separating allocation in which the high-risk agent has full consumption insurance at \( \bar{c}_b \). High-risk agents have no incentive to engage in further trading with other agents in market \( B \). This is not the case for the low-risk agents, however.

The low-risk agent purchases a contract \( \hat{c}_a = (\hat{c}_{a1}, \hat{c}_{a2}) \) that provides less than full insurance. Low-risk agents would like to “unbundle” the contract by separating its state-contingent components and trading. Suppose that a private market opens in which claims can be traded at prices \( \hat{q}_{ai} \). The agent chooses \( x_a = (x_1, x_2) \) to maximize his objective function subject to the private market budget constraint

\[
\hat{q}_{a1}\hat{c}_{a1} + \hat{q}_{a2}\hat{c}_{a2} \geq \hat{q}_{a1}x_1 + \hat{q}_{a2}x_2, \tag{8}
\]

where the left side is the private market value of the contract purchased by a type \( a \) agent in market \( A \). If only type \( a \) agents enter the original market \( A \) and the subsequent private market, then the private-market clearing prices are \( \hat{q}_{ai} = g_{ai} \) and the low risk agent would choose constant consumption \( \bar{c}_a \). At this point, there are no further gains from trade. High risk agents are no worse off because of this arrangement.

The difficulty introduced by private trading is the new allocation \( \bar{c}_a \) does not satisfy the original incentive compatibility constraints (4). Hence, to implement the incentive-efficient allocation, it is critical that agents be prevented from private trading. If private trades occur, low-risk agents are better off while high-risk agents are no worse off. But this creates the following difficulty: If high-risk agents know that low-risk agents intend to trade in private markets, then a high-risk agent may have an incentive to misrepresent his type so that he too can participate in the private market. If \( \bar{c}_a > \bar{c}_b \), then a type \( b \) agent will misrepresent his type to enter market \( A \). But if all agents self select into market \( A \), then \( \bar{c}_a \) is no longer feasible at the prices \( (g_{a1}, g_{a2}) \) and the self-selection into markets will break down.

The idea that subsequent trading opportunities can change the information revealed by an agent is studied by Krassa [1999]. He examines private information exchange economies with allocations that cannot be “improved,” in that the agent would not wish to deviate by revealing further information or by the retrading of goods. An alternative trading mechanism is considered next, in which private markets are in
equilibrium.

3 Private Trading

If a separating allocation is chosen by the social planner and if private trading among agents cannot be prevented, then agents will have an incentive to engage in private trading. Suppose that agents can trade in contingent claims markets and are price-takers in those markets. Let $q_{\eta i}$ denote the price of a contingent claim in market $\eta \in \{A, B\}$ for state $i \in \{1, 2\}$. For simplicity, normalize the prices so that $q_\eta = \frac{q_{\eta 1}}{q_{\eta 2}} \eta \in \{a, b\}$.

Markets are said to be separate if $q_\eta \neq q_h$ for $\eta, h \in \{a, b\}$, $h \neq \eta$ and there is trading in both markets. If the relative price differs across markets, then an arbitrage opportunity exists because contracts are not exclusive, so that the elimination of arbitrage profit opportunities will result in the equalization of prices across markets. The market structure that will emerge when there is private trading is discussed in section (5).

If the social planner chooses the pooling allocation, so that $c_{\eta i} = \bar{\theta}$ for all $\eta$ and $i$, then there is no incentive to enter into private trading arrangements because agents are identical across states and types. Whether or not the social planner chooses the pooling allocation will depend on the Pareto weights assigned to each type. In the discussion below, I assume that the social planner will choose a separating allocation, which is a restriction on the Pareto weights, which are parametric to the model.

3.1 Constrained Efficiency with Private Trading

Suppose that consumers are offered a menu\(^2\) of contracts $\{c(a), c(b)\} \in \mathcal{F}$ and that agents have access to a competitive market where they can trade the state-contingent components of a contract at the relative price $q$. An agent treats the menu of contracts $\{c(a), c(b)\}$ and the relative price $q$ in the private market as given.

An agent chooses his optimal reporting strategy $h \in \{a, b\}$ that determines his endowment $c(h) = (c_1(h), c_2(h))$. His actual after-trade consumption $(x_1, x_2)$ may differ from his consumption allocation $c(h)$.

\(^2\)The pooling allocation is ruled out by assumption on the Pareto weights.
An agent reporting type $h$ faces a budget set

$$\hat{B}(c(h), q) \equiv \{ x \in C \mid qx_1 + x_2 = qc_1(h) + c_2(h) \}. \quad (9)$$

Given the menu $\{c(a), c(b)\}$ and relative price $q$, a type $\eta$ agent solves

$$\hat{V}(\{c(a), c(b)\}, q; \eta) = \max_{\{h,x_1,x_2\}} g_{\eta} U(x_i) \quad (10)$$

subject to $h \in \{a, b\}$ and

$$(x_1, x_2) \in \hat{B}(c(h), q). \quad (11)$$

Let $\hat{x}_1(\{c(a), c(b)\}, q; \eta), \hat{x}_2(\{c(a), c(b)\}, q; \eta)$ and $\hat{h}(\{c(a), c(b)\}, q; \eta)$ for $\eta \in \{a, b\}$ denote the solution.

**DEFINITION:** An equilibrium in private markets, given a menu of endowments $\{c(a), c(b)\}$, consists of a relative price $q > 0$ and, for each $\eta, \eta \in \{a, b\}$ consumption allocations $(x_1(\eta), x_2(\eta))$ and a reported type $h$ such that

(i). $\hat{x}_1(\{c(a), c(b)\}, q; \eta), \hat{x}_2(\{c(a), c(b)\}, q; \eta)$ and $\hat{h}(\{c(a), c(b)\}, q; \eta)$ solve (10) subject to $h \in \{a, b\}$ and the budget constraint (11).

(ii). Markets clear:

$$\sum_{\eta} f_{\eta} \sum_{i} g_{\eta i} \hat{x}_i(\{c(a), c(b)\}, q; \eta) \leq \sum_{\eta} f_{\eta} \sum_{i} g_{\eta i} c_i(\hat{h}(\{c(a), c(b)\}, q; \eta)).$$

The existence of an equilibrium in adverse selection insurance economies can be problematic and is the topic of discussion in section (3.2).

The constrained efficient allocation with private markets and unobservable consumption, called the $SP^3$ program, or the third best program, is studied next. The social planner picks $\{c(a), c(b)\} \in F$ that maximize the Pareto-weighted expected utility of agents subject to the feasibility condition, the incentive compatibility constraints and the possibility that agents may trade in private markets. Let $\Psi_\eta$ denote a Pareto weight for a type $\eta$ agent such that $\Psi_\eta > 0$ and $1 = \Psi_a + \Psi_b$. The constrained efficient social planning problem with private trading ($SP^3$) is

$$\max_{\{c(a), c(b)\}} \sum_{\eta} \Psi_\eta \sum_{i} g_{\eta i} U(c_i(\eta)) \quad (12)$$

subject to

$$\tilde{\theta} \geq \sum_{\eta} f_\eta \sum_{i} g_{\eta i} c_i(\eta), \quad (13)$$

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\[ \sum_i g_{ai} U(c_i(a)) \geq \hat{V}(\{c(a), c(b)\}, q; a), \]  
(14) \[ \sum_i g_{bi} U(c_i(b)) \geq \hat{V}(\{c(a), c(b)\}, q; b). \]  
(15)

The first constraint (13) is feasibility of the allocation. The second and third constraints require the consumption allocation to result in expected utility that is at least as high as the agent type \( \eta \) can achieve by reporting he is type \( \hat{h}(\{c(a), c(b)\}, q; \eta) \) and then trading in private markets at relative price \( q \). It is straightforward to show that, if for some \( h \in \{a, b\} \),

\[ qc_1(h) + c_2(h) > qc_1(\eta) + c_2(\eta) \quad \eta, h \in \{a, b\} \quad h \neq \eta, \]

then every agent will announce that he is type \( h \) or \( h = \hat{h}(\{c(a), c(b)\}, q; a) \hat{h}(\{c(a), c(b)\}, q; b) \). This follows\( ^3 \) because the indirect utility function \( \hat{V} \) is strictly increasing in the market value of the consumption contract and all agents face the identical relative price in private markets. As a result, all agents will then face the identical budget set \( \hat{B}(c(h), q) \). It follows that the consumption \( x \in \hat{B}(c(h), q) \) will be incentive compatible.

Define the value of the consumption allocation \( c(h) \) in private markets as

\[ (1 + q)w \equiv qc_1(a) + c_2(a) = qc_1(b) + c_2(b), \]

where \( w \) is the certain consumption endowment, so that the endowment is \( (w, w) \).\(^3\) I now show the conditions under which choosing the consumption allocation to solve \((SP3)\) is equivalent to the social planner choosing \((w, q)\).

If a type \( \eta \) agent has the endowment \( w \) and faces relative price \( q \) in private markets, his budget set is

\[ B(w, q) \equiv \{ x \in C \mid (1 + q)w = qx_1 + x_2 \}. \]

The agent solves

\[ V(w, q; \eta) = \max_{\{x \in B(w, q)\}} \sum_i g_{\eta i} U(x_i) \]  
(16)

\(^3\)Alternatively one can define \( \hat{w} \) as

\[ \hat{w} = qc_1(h) + c_2(h) \]

so that

\[ \hat{w} = (1 + q)w. \]

Specifying \((\hat{w}, q)\) is equivalent to specifying \((w(h), q)\). The problem is posed using \( w \) because the opportunity set for \( w \) is straightforward.
Denote the solution as $\xi_\eta(w, q) \equiv (\xi_{\eta 1}(w, q), \xi_{\eta 2}(w, q))$.

The social planning problem can now be restated as follows. The social planner picks $0 < w \leq \bar{\theta}$ and relative price $q > 0$ to solve

\[
\max_{\{w, q\}} \sum_\eta \Psi_\eta V(w, q; \eta) \quad (17)
\]

subject to

\[
\bar{\theta} \geq \sum_\eta f_\eta \sum_i g_{\eta i} \xi_{\eta i}(w, q).
\]

Let $(w^*, q^*)$ denote the solution.

I now prove the equivalence between the solution to (17) and the solution to ($SP_3$).

**Lemma 1** Let $(w^*, q^*)$ be the solution to (17) and $\xi_\eta(w^*, q^*)$ for $\eta \in \{a, b\}$ be solutions to (16). Then

\[
c_i(\eta) = \xi_{\eta i}(w^*, q^*) \quad \text{for} \quad \forall \quad \eta \in \{a, b\} \quad \text{and} \quad \forall \quad i \in \{1, 2\} \quad (19)
\]

is a solution to ($SP_3$).

**Proof.**

The first step is to show the solution to $SP_3$, denoted $(c^*(a), c^*(b))$ can be implemented for some $w$ and $q$ satisfying the feasibility condition for (18) in that $(c^*(a), c^*(b))$ would solve (17). The second step is to take $(w^*, q^*)$ solving (17) and show the $(c(a), c(b))$ given by

\[
c_i(\eta) = \xi_{\eta i}(w^*, q^*; \eta)
\]

are feasible and solve ($SP_3$).

Take any solution $\{c(a), c(b)\}$ to $SP_3$ and let $q$ be the equilibrium relative price for private markets given these allocations. Note that the incentive compatibility constraint for $SP_3$ can be written

\[
\sum_i g_{\eta i} U(c_i(\eta)) \geq V(qc_1(h) + c_2(h), q; \eta) \quad \text{for} \quad \forall h, \eta
\]

meaning that type $\eta$ doesn’t get more expected utility from the endowment by pretending to be type $h$ and trading. By (17)

\[
V(qc_1(\eta) + c_2(\eta), q; \eta) \geq V(qc_1(h) + c_2(h), q; \eta)
\]
Since $V$ is strictly increasing in its first argument, this is equivalent to

$$qc_1(\eta) + c_2(\eta) = \max_{h \in \{a, b\}} [qc_1(h) + c_2(h)]$$

The next step is to show that for $(w, q)$ and the given allocation $\{c(a), c(b)\}$, will solve (17) for any $\eta$ and this implies (18) if $x_{i\eta}(w, q) = c_i(\eta)$ for all $\eta$ and $i$ because $\{c(a), c(b)\}$ satisfies (12). But $\{c(a), c(b)\}$ is a solution to (10) and is still a solution under the additional constraint $\eta = h$. This proves that any solution $\{c(a), c(b)\}$ to (12) may be implemented through the appropriate choice of $w$ and $q$. Moreover the values of the maximand in (12) and (17) are equal for these parameter values; hence the maximum in (17) is at least as large as the one in (12).

Suppose now that $(w^*, q^*)$ solve problem (17); let $\{c(a), c(b)\}$ be given by $c_i(\eta) = \xi_{i\eta}(w^*, q^*)$ where $\{\xi_a(w^*, q^*), \xi_b(w^*, q^*)\}$ is the solution to (16) for $(w^*, q^*)$. The next step is to check that $\{c(a), c(b)\}$ is feasible for (17) so it satisfies (13)–(15). Clearly (13) follows from (18). Since the constraint to (17) is binding

$$q^*c_1(a) + c_2(a) = q^*c_1(b) + c_2(b) = (1 + q^*)w^*$$

so that

$$\hat{V}(\{c(a), c(b)\}, q^*; \eta) = V(w^*, q^*; \eta) \quad \text{for all} \quad \eta \in \{a, b\}.$$  

To be completed

The two approaches to solving for the constrained efficient allocation with private markets are illustrated in Figure (1). The social planner can choose consumption allocations $\{c(a), c(b)\}$ such that $c(\eta) \in B(w, q)$. In Figure (1), the average feasibility constraint is the line through the point $\bar{\theta}$ with slope $-\bar{R}$, with a horizontal intercept at $\bar{\theta}_1$ and vertical intercept at $\bar{\theta}_2$. The budget constraint for an agent is the line through the point $w$ on the 45-degree line with slope $-q$. The budget constraint intersects the resource constraint at the point $S$ where $S = (\hat{\theta}_1, \hat{\theta}_2)$. The endowment point is $E = (\theta_1, \theta_2)$ (the autarky point). The consumption allocations $c(a)$ and $c(b)$ are in the budget set and satisfy feasibility. The social planning problem $SP3$ specifies the

$\bar{\theta} = p_1\hat{\theta}_1 + p_2\hat{\theta}_2$

$$(1 + q)w = q\hat{\theta}_1 + \hat{\theta}_2.$$
Figure 1: Budget Set with Private Trading
consumption allocations $c(a), c(b)$. Alternatively, the social planner can be modeled as specifying a point $w$ on the 45-degree line and the slope $q$.\(^5\)

Before solving the social planner’s problem, it is critical to establish the conditions under which a competitive equilibrium in private markets exists. As discussed in Bisin and Gottardi [2006] and Labadie [2009], the existence of a competitive equilibrium can be problematic.

### 3.2 Equilibria in Private Trading

Conditions that ensure the existence of an equilibrium are discussed in this section when the social planner picks $w \in W \equiv (0, \bar{\theta})$. The solution $\xi_\eta(w, q)$ to (16) satisfies the first-order condition

$$
\frac{U''(\xi_{a1}(w, q))}{R_\eta} = q.
$$

Since $U$ is strictly concave and satisfies the Inada conditions, it is immediate that the solution $\xi_\eta(w, q)$, $\eta = a, b$ exists and is unique. Moreover $\xi_\eta(w, q)$ is continuous and strictly increasing in $w$ and strictly decreasing in $q$. Since both agents have identical budget sets $B(w, q)$, the demand functions $\xi_\eta(w, q)$ are incentive compatible for any $(w, q)$. It follows from monotonicity that the demand functions have the following properties.

(i) Since $R_b > R_a$, high-risk agents (type b) purchase more consumption in state 1 than low-risk agents (type a),

$$
\xi_{b1}(q, w) > \xi_{a1}(q, w) \quad \text{for all } q > 0 \quad \text{and } 0 < w < \bar{\theta}
$$

(ii) A type $\eta$ agent insures partially (fully, more than fully) as $q > R_\eta$ ($q = R_\eta, q < R_\eta$)

$$
\xi_{a1}(w, q) \leq w \quad \text{as } q \gtrless R_\eta
$$

**Definition:** A *private trading equilibrium* is an initial endowment $w \in W$, a price $q^e > 0$ and consumption demands $(x_a^e, x_b^e) \in \mathcal{F}$, such that

(i). Agents solve (16), so that

$$
x_\eta^e = \xi_\eta(w, q^e) \quad \text{for } \eta = a, b.
$$

\(^5\)The social planner can also choose to reallocate the endowment by specifying $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2)$ and the price $q$. The discussion below focuses on the equivalence between the solution to SP3 and the social planning problem with decision variables $(w, q)$, discussing the equivalence of other approaches with the SP3 in an appendix (to be written).
(ii). The equilibrium condition is satisfied at $q^e$

$$\bar{\theta} = \sum_{\eta} f_{\eta} \sum_i g_{\eta i}(w, q^e).$$

The existence of an equilibrium can be established under certain conditions as follows: For any $q > 0$ and $w \in W$, the equilibrium condition can be expressed as

$$\bar{\theta} = \sum_{\eta} f_{\eta} \{g_{\eta 1}\xi_1(w, q) + g_{\eta 2}[(1 + q)w - q\xi_1(w, q)]\}$$

or

$$\bar{\theta} - (1 + q)p_2w = \sum_{\eta} f_{\eta}g_{\eta 2}\xi_1(w, q)[R_\eta - q].$$

Define

$$H(w, q) \equiv \bar{\theta} - p_2(1 + q)w,$$

and define $\Xi_\eta : W \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\Xi_\eta(w, q) \equiv g_{\eta 2}\xi_1(w, q)[R_\eta - q]$$

and let

$$\Xi(w, q) \equiv \sum_{\eta} f_{\eta}\Xi_\eta(w, q).$$

The key equation (21) can then be expressed as

$$H(w, q) - \Xi(w, q) = 0.$$  \hspace{1cm} (23)

The properties of the functions $H$ and $\Xi$ are described next.

The function $H$ has the following properties.

(i). $H$ is decreasing in $q, w$.

(ii). At $q = 0$, $H(w, 0) = \bar{\theta} - p_2w$.

(iii). As $q$ increases, $\lim_{q \to \infty} H(w, q) = -\infty$.

(iv) At $w = \bar{\theta}$, $H(\bar{\theta}, q) = \bar{\theta}p_2[R - q]$.

(v). There exists a value of $q$ given $w \in W$ such that $H(w, q) = 0$. Define $\tilde{q} : W \to \mathbb{R}^+$ as

$$\tilde{q}(w) \equiv \left[\frac{\bar{\theta} - p_2w}{p_2w}\right].$$

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Observe that $\bar{q}$ is decreasing in $w$. At $w = \bar{\theta}$, $\bar{q}(\bar{\theta}) = \bar{R}$. Also

$$\lim_{w \to 0} \bar{q}(w) = \infty.$$ 

The function $H$ can be rewritten as

$$H(w, q) = p_2 \left[ \frac{\bar{\theta}}{p_2} - (1 + q)w \right].$$

In figure (1), the function $H$ is proportional to vertical distance between the intercept of the resource constraint $\frac{\bar{\theta}}{p_2}$ and the vertical intercept of the budget constraint $(1 + q)w$.

Next, the properties of the function $\Xi$ depend on the demand functions $\xi$. Observe that

$$\lim_{q \to 0} \xi_{\eta_1}(w, q) = +\infty \quad \text{and} \quad \lim_{q \to 0} \xi_{\eta_2}(w, q) = 0, \quad \eta = a, b,$$

$$\lim_{q \to +\infty} \xi_{\eta_1}(w, q) = 0 \quad \text{and} \quad \lim_{q \to +\infty} \xi_{\eta_2}(w, q) = +\infty, \quad \eta = a, b.$$

It follows immediately that

$$\lim_{q \to 0} \Xi(w, q) = +\infty$$

and

$$\lim_{q \to +\infty} \Xi(w, q) = 0.$$

**Lemma 2** Given $w \in W$, there exists a $\bar{R} < q < R_b$ such that

$$\Xi(w, q) = 0.$$

**Proof.**

At $q = \bar{R}$, $\Xi(w, \bar{R}) > 0$. This follows because

$$\Xi(w, q) = f_a g_a \xi_{a_1}(w, q)[R_a - \bar{R}] + f_b g_b \xi_{b_1}(w, q)[R_b - \bar{R}]$$

$$> f_a g_a \xi_{a_1}(w, q)[R_a - \bar{R}] + f_b g_b w[R_b - \bar{R}]$$

$$> f_a g_a w[R_a - \bar{R}] + f_b g_b w[R_b - \bar{R}] = 0$$

where the first inequality follows because $\xi_{b_1}(w, \bar{R}) > w$ and the second inequality follows because $\xi_{a_1}(w, \bar{R}) < w$. Hence $\Xi(w, \bar{R}) > 0$. 

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At \( q = R_b \),
\[
\Xi(w, R_b) = f_a g_a 2 \xi_a (w, R_b) |R_a - R_b| < 0.
\]
Hence, because \( \Xi \) is continuous, there exists a \( q \) such that \( \bar{R} < q < R_b \) satisfying
\[
\Xi(w, q) = 0.
\]

Given \( w \in W \), define the solution to this equation or
\[
\Xi(w, q) = 0.
\]
as \( \hat{q}(w) \).

There are three cases: (i) \( \hat{q}(w) < \tilde{q}(w) \), (ii) \( \hat{q}(w) > \tilde{q}(w) \), and (iii) \( \hat{q}(w) = \tilde{q}(w) \).

**Lemma 3** Let \( w \in W \). If \( \hat{q}(w) < \tilde{q}(w) \), then there exists a \( q \) satisfying (23) such that \( q^e \in (0, \hat{q}(w)) \cup (\tilde{q}(w), \infty) \).

**Proof.**
It follows from assumptions (1)-(2) that \( \Xi(w, q) \) is continuous and continuously differentiable in \( q \). At \( q = 0 \),
\[
\Xi(w, 0) > H(w, 0).
\]
Since \( \Xi(w, \hat{q}(w)) < H(w, \hat{q}(w)) \), it follows that there is a \( q \leq \hat{q}(w) \) such that (23) holds because
\[
H(w, \hat{q}(w)) > H(w, \hat{q}(w)) = 0.
\]
Since \( \Xi(w, \hat{q}(w)) < H(w, \hat{q}(w)) = 0 \), and \( \lim_{q \to -\infty} \Xi(w, q) = 0 \) while \( \lim_{q \to -\infty} H(w, q) = -\infty \), it follows that there is some \( q > \hat{q}(w) \) such that (23) holds.

At \( q^e = \tilde{q}(w) \), \( \Xi(w, \tilde{q}(w)) < 0 = H(w, \hat{q}(w)) \). Since \( \lim_{q \to -\infty} H(w, q) = -\infty \) and \( \lim_{q \to -\infty} \Xi(w, q) = 0 \), it follow that there is some \( q^e > \tilde{q}(w) \) such that (23) holds.

An example is provided in Figure 2.

**Lemma 4** If \( \hat{q}(w) \geq R_b \), then \( w \leq w(R_b) \) where
\[
w(R_b) = \frac{\tilde{\theta}}{p_2(1 + R_b)}.
\]
Figure 2: Determination of the equilibrium price

Figure 3: Example where an equilibrium does not exist
Proof.

By definition $\tilde{q}(W(R_b)) = R_b$ and $\tilde{q}(w)$ is strictly decreasing in $w$.

An example where an equilibrium does not exist is illustrated in the next figure.

**Lemma 5** If $\hat{q}(w) = \tilde{q}(w)$, then $w > w(R_b)$.

Proof.

This follows because $\bar{R} < \hat{q}(w) < R_b$ and for any $0 < w < w(R_b)$, $\hat{q}(w) \geq R_b$. Since $\tilde{q}(w)$ is strictly decreasing in $w$, it follows that $w > w(R_b)$.

The pair $(w, q)$ such that $\hat{q}(w) = \tilde{q}(w)$ will satisfy the pair of equations

\[ \Xi(w, q) = 0 \]
\[ H(w, q) = 0 \]

or $q$ is the solution to

\[ \Xi \left( \frac{\tilde{\theta}}{p_2(1 + q)}, q \right) = 0 \]

The function $\Xi$ is not monotonic in $w$ or $q$. If a solution exists, denote it as $q^*$ and define

\[ w^* = \frac{\tilde{\theta}}{p_2(1 + q^*)}. \]

**Lemma 6** If $\hat{q}(w) = \tilde{q}(w)$, then there is an equilibrium price $q \in (\bar{R}, R_b)$.

Proof.

The lower bound on the value of $\tilde{q}(w)$ is at $\bar{R}$ where $w = \tilde{\theta}$. If $w < \tilde{\theta}$ then $\tilde{(q)(w)} > \bar{R}$. Since $\hat{q}(w) < R_b$, the inequality follows.

The results for the existence of an equilibrium are summarized in the following theorem.

**Theorem 1** (i) If $0 < w \leq w(R_b)$, then there exits a $q^e$ solving (23) such that $q^e \in (0, \hat{q}(w)) \cup (\tilde{q}(w), \infty)$.

(ii) If $w \geq w(R_b)$ and $\hat{q} < \tilde{q}(w)$, then there exists an equilibrium $q^e$ such that $q^e \in (0, \tilde{q}(w)) \cup (\hat{q}(w), \infty)$.

(iii) If $w^e$ exists and $w^e > w \geq w(R_b)$ and $\hat{q}(w) = \tilde{q}(w)$, then $q^e = \hat{q}(w) = \tilde{q}(w)$ is an equilibrium.

(iv) If $\tilde{\theta} > w > w^*$ such that $\hat{q} > \tilde{q}(w)$, then if an equilibrium exists, $q^e > \hat{q}(w)$.

Proof to be completed.
3.3 Solution of the Constrained Problem with Private Trading

The first-order conditions are

\[ \sum \Phi \frac{\partial V(w, q; \eta)}{\partial w} = \lambda \sum f \sum g \frac{\partial \xi_i(w, q)}{\partial w} \]  

(25)

\[ \sum \Phi \frac{\partial V(w, q; \eta)}{\partial q} = \lambda \sum f \sum g \frac{\partial \xi_i(w, q)}{\partial q} \]  

(26)

Solve (26) for \( \lambda \) and substitute into (25) and rewrite

\[ \left( \sum \Phi \frac{\partial V(w, q; \eta)}{\partial w} \right) \left( \sum f \sum g \frac{\partial \xi_i(w, q)}{\partial q} \right) = \left( \sum \Phi \frac{\partial V(w, q; \eta)}{\partial q} \right) \left( \sum f \sum g \frac{\partial \xi_i(w, q)}{\partial w} \right) \]  

(27)

Roy’s identity, which is

\[ \frac{\partial V(\eta, q)}{\partial q} = -\frac{\partial V(\eta, w)}{\partial w} \xi(\eta, q), \]

can be substituted into (28) to obtain

\[ \left( \sum \Phi \frac{\partial V(w, q; \eta)}{\partial w} \right) \left( \sum f \sum g \frac{\partial \xi_i(w, q)}{\partial q} \right) = \left( -\sum \Phi \frac{\partial V(\eta, w)}{\partial w} \xi(\eta, q) \right) \left( \sum f \sum g \frac{\partial \xi_i(w, q)}{\partial w} \right) \]  

(28)

The Slutsky-Hicks equations are

\[ \frac{\partial h_1(q, U^*_\eta)}{\partial q} = \frac{\partial \xi_1(w, q)}{\partial q} + \frac{\partial \xi_1(w, q)}{\partial w} \xi_1(w, q) \]

\[ \frac{\partial h_2(q, U^*_\eta)}{\partial q} = \frac{\partial \xi_2(w, q)}{\partial q} + \frac{\partial \xi_2(w, q)}{\partial w} \xi_1(w, q) \]

where \( h_\eta(q, U^*_\eta) \) is the Hicksian or compensated demand curve and \( U^*_\eta \) is a given level of utility. Solve the Slutsky-Hicks equation for \( \frac{\partial \xi_i(w, q)}{\partial w} \) and substitute into (28) to obtain

\[ \left( \sum \Phi \frac{\partial V(w, q; \eta)}{\partial w} \right) \left( \sum f \sum g \frac{\partial h_\eta(q, U^*_\eta)}{\partial q} \right) \]

(29)

\[ = \left( \sum \Phi \frac{\partial V(w, q; \eta)}{\partial w} \right) \sum f \xi_1(w, q) \sum_i g \frac{\partial \xi_i(w, q)}{\partial w} - \left( \sum \Phi \xi_1(w, q) \frac{\partial V(w, q; \eta)}{\partial w} \right) \sum f \sum_i g \frac{\partial \xi_i(w, q)}{\partial w} \]

4 Logarithmic Example

When \( U(c) = \ln c \), the demand functions are

\[ \xi_{\eta_1} = \frac{\hat{w}g_{\eta_1}}{q} \]
\[ c_{\eta 2} = g_{\eta 2} \hat{w} \]

where \( \hat{w} \equiv (1 + q)w \). The social planner picks \( \hat{w}, q \) to maximize

\[ \sum_{\eta} \Phi_\eta \left[ g_{\eta 1} \ln \left( \frac{g_{\eta 1} \hat{w}}{q} \right) + g_{\eta 2} \ln (g_{\eta 2} \hat{w}) \right] \tag{30} \]

subject to the resource constraint

\[ \bar{\theta} \geq \sum_{\eta} f_\eta \left[ \frac{g_{\eta 1} \hat{w}}{q} + g_{\eta 2} \hat{w} \right] \]

Let \( \mu \) denote the multiplier for the resource constraint. The first-order conditions with respect to \( \hat{w}, q \) are

\[ \sum_{\eta} \Phi_\eta \frac{1}{\hat{w}} = \mu \left[ \sum_{\eta} f_\eta \left( \frac{g_{\eta 1}^2}{q} + g_{\eta 2}^2 \right) \right] \tag{31} \]

\[ \sum_{\eta} \Phi_\eta \frac{g_{\eta 1}}{q} = \mu \hat{w} \sum_{\eta} f_\eta \frac{g_{\eta 1}^2}{q^2} \tag{32} \]

Solve (31) for \( \mu \), substitute into (32), and solve for \( q^* \)

\[ q^* = \left[ \frac{\sum_{\eta} f_\eta g_{\eta 1}^2 [1 - \sum P_{hi\eta} g_{\eta 1}]}{\sum \Phi_\eta g_{\eta 1} \sum f_\eta g_{\eta 2}^2} \right] \tag{33} \]

The value of \( w \) is determined from

\[ w^* = \left( \frac{\bar{\theta}}{1 + q^*} \right) \left[ \sum_{\eta} f_\eta \left( \frac{g_{\eta 1}^2}{q^*} + g_{\eta 2}^2 \right) \right]^{-1} \]

Figure 3 illustrates the optimal solution for the following parameter values: \( f_a = 0.6, g_{a1} = 0.3, g_{b1} = 0.6 \) and \( \Phi_a = 0.75 \). The endowment point (autarky) is \( E = [2, 8] \) and \( \bar{\theta} = 5.48 \). The equilibrium price is \( q^e = 0.92 \), which lies between \( \bar{R} = 0.72 \) and \( R_b = 1.5 \), and the optimal level of income is \( w = 4.98 \). In this example, the social planner can achieve the optimal consumption allocation \( c(a) = [3.11, 6.70] \) and \( c(b) = [6.23, 3.83] \) by changing the initial endowment to \( S = [0.60, 9.01] \) or by offering the menu of consumption \( \{c(a), c(b)\} \), or by setting \( w = 4.98 \). The functions \( H \) and \( \Xi \) for the optimal level of income \( w^* \) are illustrated in the Figure (5).

### 4.1 Other Numerical Examples

In the example illustrated in Figure (6), \( \Phi_a = f_a \), \( g_{a1} = 0.3, g_{b1} = 0.7 \), \( \bar{R} = 0.72 \), \( \bar{\theta} = 5.48 \). The solution to the social planner’s problem is \( q^e = 0.76 \) and \( w = 5.03 \). Observe that the budget constraint does not intersect the average feasibility constraint (fair odds line).
5 Market Structure

There are two possibilities for the organization of markets: there is a single market in which all claims contingent on realization $i \in \{1, 2\}$ trade for the identical price $q_i$ or else there are separate markets $A$ and $B$ where contingent claims trade at prices $q_{\eta i}$. The social planner offers a consumption vector $c \in I$. An agent who announces he is type $\eta$ receives $c_\eta$. If the agent trades in market $h$, $h \in \{A, B\}$, he faces a budget constraint

$$q_{h1}c_{\eta 1} + q_{h2}c_{\eta 2} = q_{h1}x_{\eta 1} + q_{h2}x_{\eta 2} \quad h \in \{a, b\}$$

Nonexclusivity of contracts implies that agents can enter into multiple trades and that current trades are not conditional on previous trades. The first issue is whether markets can be separate when there is private information about type and nonexclusivity in contracts. An agent can enter into contingent claims markets multiple times as long as the budget constraint is satisfied.

There are three possibilities for $c \in I$: (i) both types have full insurance; (ii) one type has full insurance while the other does not; (iii) neither type has full insurance. To start, suppose there are separate markets.
Figure 5: The functions $H$ and $\Xi$
If an agent of type $\eta$ self selects into market $\eta$, then by symmetry all agents of type $\eta$ self select into market $\eta$. The implications of separate markets are derived for the three cases.

**Case i:** Both types have full insurance at consumption levels $\bar{c}_a, \bar{c}_b$. If $\bar{c}_a \neq \bar{c}_b$, then both agents will announce the type that provides the highest level of certain consumption, so incentive compatibility requires $\bar{c}_a = \bar{c}_b$, which is feasible if $\bar{c}_\eta = \bar{\theta}$ (if less than $\bar{\theta}$ there are unused resources). If the social planner offers each agent $\bar{\theta}$ for both types and states, then agents are identical across all states and types so there is no idiosyncratic risk, no trading and hence no side markets are formed.

**Case ii:** One type has full insurance at $\bar{c}_\eta$ while the other does not, so that $c_{h1} \neq c_{h2}, h \neq \eta$. Assume that $\bar{c}_\eta \neq \bar{\theta}$. If only type $h$ agents self select into market $h$, then equilibrium prices satisfy $q_{hi} = g_{hi}$ and type $h$ agents will trade such that they achieve full consumption insurance equal to

$$\hat{c}_h = g_{h1}c_{h1} + g_{h2}c_{h2}.$$ 

If $\hat{c}_h \neq \hat{c}_\eta$, then the final consumption allocation is not incentive compatible. If, as assumed, a type $h$ agent self-selects into market $h$, then $\hat{c}_h \geq \hat{c}_\eta$, otherwise he would have announced type $\eta$. It follows that, if
type $h$ agents obtain $\hat{c}_h > \hat{c}_\eta$, then type $\eta$ agents would also announce type $h$ in anticipation of the higher consumption after trade and the equilibrium price cannot satisfy $q_{hi} = g_{hi}$. If $\hat{c}_\eta = \tilde{\theta}$, then after trading with other agents of type $h \neq \eta$, a type $h$ agent has final consumption $\hat{c}_h = \tilde{\theta}$. Type $\eta$ agents have no incentive to announce they are type $h$ and type $h$ agents are indifferent between announcing type $\eta$ and type $h$.\(^6\)

**Case iii:** Neither agent has full insurance. There are two sub-cases. If $c_i \equiv c_{\eta i} = c_{hi}$, as would be the case if agents receive identical contingent endowments $\theta_1, \theta_2$, and if markets were separate, then all agents will engage in arbitrage trading because

$$0 \neq c_1 [q_{a1} - q_{b1}] + c_2 [q_{a2} - q_{b2}]$$

and an arbitrage opportunity exists. Trading across markets will eliminate the arbitrage opportunity until $q_{\eta i} = q_{hi}$. In the other case, if agents receive different endowments, $c_{ai} \neq c_{bi}$, then agents will engage in trade to eliminate arbitrage profits. Moreover, if the value of the consumption allocations $c \in I$ in the side markets is different,

$$q_1 c_{a1} + q_2 c_{a2} \neq q_1 c_{b1} + q_2 c_{b2}$$

then an agent will announce the type with the highest value of consumption in side markets.

To summarize, unless one or both types have full consumption insurance where certain consumption is equal to $\tilde{\theta}$, there will be a single market for contingent claims. Moreover, if the allocation $c \in I$ has the property that neither type has full insurance, then the consumption allocation for each type must have the same value in side markets. Specifically, define

$$q \equiv \frac{q_1}{q_2}$$

then the requirement that allocations have the identical market value requires

$$w \equiv q c_{a1} + c_{a2} = q c_{b1} + c_{b2}$$

Define $B(c_\eta, q)$ to be the set of consumption trades that satisfy the budget constraint (with equality) for relative price $q$, and endowment $c_\eta$

$$B(c_\eta, q) \equiv \{x \in C \mid qx_1 + x_2 = ac_{\eta 1} + c_{\eta 2}\}.$$  \hfill (35)

\(^6\)Assume that if an agent is indifferent, then the agent will truthfully reveal his type.
Hence the opportunity to trade in side markets imposes an additional restriction on consumption allocations besides incentive compatibility, namely that allocations must also have the same market value.

6 Conclusion

An exchange economy with adverse selection and private information is studied under the assumption that risk averse agents trade directly in a contingent claims market. Markets are not separated by type, contracts are not exclusive, and agents can enter into side arrangements. The result is the competitive equilibrium with anonymity. Anonymity means that all agents face the same budget set in net trades. The allocation is individually incentive compatible, although it is not incentive-efficient. Since agents face the same budget constraint in net trades, but face different endowment distribution risk, agents will execute different net trades depending on type. The result that some states are under-insured while others are over-insured. The equilibrium has the property that the marginal rate of substitution is equalized across states in the contingent claims market. The anonymous equilibrium is derived and shown to exist, although it may not be unique. A closed-form solution is derived for a set of parameter values and plotted in Figure 3. A comparison of expected utility for various allocations is provided in Table I.

One direction for future work is to determine the conditions under which the anonymous mechanism is coalitionally incentive compatible and to determine whether the simple market mechanism studied here is an organizational structure that will emerge endogenously using coalition theory, as in Boyd, Prescott and Smith [2].
References